

The double copy: gravity from gluons

C. D. White^a

^aCentre for Research in String Theory, School of Physics and Astronomy, Queen Mary University of London, 327 Mile End Road, London E1 4NS, UK

ARTICLE HISTORY

Compiled November 27, 2017

ABSTRACT

Three of the four fundamental forces in nature are described by so-called *gauge theories*, which include the effects of both relativity and quantum mechanics. Gravity, on the other hand, is described by General Relativity, and the lack of a well-behaved quantum theory - believed to be relevant at the centre of black holes, and at the Big Bang itself - remains a notorious unsolved problem. Recently a new correspondence, the *double copy*, has been discovered between scattering amplitudes (quantities related to the probability for particles to interact) in gravity, and their gauge theory counterparts. This has subsequently been extended to other quantities, providing gauge theory analogues of e.g. black holes. We here review current research on the double copy, and describe some possible applications.

KEYWORDS

Gauge theories, gravity, string theory, particle physics.

1. Introduction

Fundamental physics continues to seek answers to the biggest questions facing humankind: where did the universe come from? What is it made of? How will it end? The past few hundreds of years of development have combined precision experiments with abstract theoretical reasoning, and our current understanding of the basic building blocks of nature is as follows. The universe contains *matter*, which is acted upon by *forces*. All observed forces are consequences of only four *fundamental forces*. Three of these - electromagnetism, the weak nuclear force, and the strong nuclear force - are described by the Standard Model (SM) of particle physics, which also lists all known matter particles – quarks and leptons – that combine to make the various composite particles observed in nature. All forces and matter types in the SM are described by fields filling all of spacetime. The equations describing these fields include the effects of quantum mechanics and relativity, so that the SM is an example of a *quantum field theory* (QFT). Indeed, the particles themselves emerge as quanta of the fields, with the canonical example being the photon (a quantum of the electromagnetic field). Furthermore, the fields have certain abstract mathematical symmetries, so that this type of QFT is also called a *(non)-Abelian gauge theory*. We will define these terms more precisely in what follows.

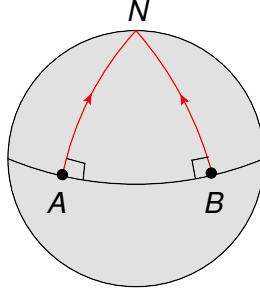


Figure 1. On a (spherical) curved space, observers at A and B are told to walk towards the north pole (N), and start off mutually parallel. Due to the curvature, the observers will move towards each other, which looks like an attractive force.

The fourth force in nature is gravity, and is currently best described by the General Theory of Relativity (GR). This tells us that space and time are dynamical, rather than being a fixed background upon which particles and forces operate. The equations of GR tell us that matter and energy warp the fabric of spacetime in a prescribed way, such that this curvature can be identified with the gravitational force. A simple analogy for how curvature can produce an attractive force is shown in figure 1. GR underpins much of our understanding of how the universe works at very large scales. It tells us that black holes may exist, for example, and gives rise to the possibility that the universe may have expanded outwards from a “big bang” at some finite time in the past. It also predicts the existence of *gravitational waves*: ripples in the fabric of spacetime that are analogous to the wave-like solutions of Maxwell’s equations for electromagnetism, and which have only recently been observed for the first time [1]. On a more practical level, one must account for the curvature of spacetime in satellite communications, and GR has even become part of everyday life through the use of positioning systems in smartphones.

Despite the wide-ranging success of both the SM and GR, many puzzles remain. It is not known, for example, why there are only four fundamental forces, and why the various matter particles have their particular properties (e.g. masses, and charges with respect to each force). It is not fully understood why matter dominated over antimatter in the early universe, and present day astrophysical observations imply the existence of both *dark matter* and *dark energy*, neither of which is present in the SM. Furthermore, the classical theory of GR breaks down at extreme points in spacetime, such as at the centre of a black hole, or the big bang itself. In such places, the curvature of spacetime becomes infinite, which is not physically sensible. It is thus widely thought that both the SM and GR are part of some larger theoretical framework, which may include quantum effects for the gravitational force, in line with the other forces. One may attempt to turn GR into a quantum field theory, such that gravity is carried by a *graviton*, analogous to the photon that carries the electromagnetic force in the SM. Such attempts fail, however, due to the fact that calculations become infinite when gravitons are emitted and absorbed at arbitrarily short distances, corresponding to high energy scales. These so-called *ultraviolet (UV) divergences* are also present in the SM, but can be removed by redefinitions of the fields and parameters entering the theory, such as particle masses, and the *coupling constants* describing the intrinsic strengths of each force. This is known as *renormalisation*, and the same procedure does not work for GR, such that the theory is described as *non-renormalisable*.

The apparent inability to reconcile quantum mechanics with the theory of gravity

is one of the most notorious open problems in theoretical physics. Quantum gravity may resolve the inability of GR to fully describe black hole physics, the big bang, or dark energy. There are a number of possibilities for such a theory. Firstly, it may turn out that GR is alright after all if thought about in the right way e.g. calculations that rely purely on *perturbation theory* (an expansion in the strength of the gravitational force) may be insufficient. One such proposal is the asymptotic safety idea of ref. [2]. Secondly, there may be a modified field theory of gravity, that is renormalisable in perturbation theory. An open possibility is $\mathcal{N} = 8$ Supergravity, which has additional matter content alongside the graviton, in a highly constrained way such that a certain symmetry between bosons and fermions (*supersymmetry*) is made manifest. Thirdly, quantum gravity may not be a field theory that gives rise to particles, but a different type of theory. One such proposal is *string theory*, in which both gauge theories and gravity emerge from the dynamics of vibrating strings. Of course, a fourth possibility is that there is no quantum theory of gravity at all, but this would leave significant unsolved puzzles in both the SM and GR.

One of the main problems in investigating quantum gravity is the sheer complexity of the calculations involved, which quickly become intractable even with the aid of powerful computers. It seems that new methods and insights are needed, and one such technique has arisen in the last few years: *the double copy* [3,4]. This relates quantities calculated in a gauge theory, with similar quantities obtained in a gravity theory. The original form of this correspondence involved scattering amplitudes - complex-valued functions of momenta which are related to the probability for a given set of particles to interact. However, it has since been extended to other types of object in gauge and gravity theories - such as exact classical solutions, including black holes. Furthermore, similar correspondences have been found between other types of field theory, with or without supersymmetry.

The double copy has the potential to revolutionise our understanding of gravity, given that it relates theories like those in the SM, whose quantum behaviour we understand well, with gravity. It suggests that, if one thinks about gravity in the right way, it is much simpler than traditional calculations in GR would seem to suggest. We also get new insights into gauge theory itself. For the double copy between scattering amplitudes in gauge and gravity theories to work, the gauge theory results have to be written to obey an intriguing symmetry between the parts relating to the charges of each gluon, and the parts relating to their momenta, polarisations etc. This is known as *Bern-Carrasco-Johansson (BCJ) duality* [5], and implies that these various degrees of freedom are much more closely related than previously thought. In order to be able to define the double copy more formally, let us first study gauge theories in more detail.

2. Gauge theories

Here, we review the basic ideas underlying gauge theories - the type of quantum field theory that describes three out of the four fundamental forces in nature, as contained in the Standard Model of particle physics. Arguably the most familiar gauge theory is that of electrodynamics, which we turn to first.

2.1. Quantum electrodynamics (QED)

The basic idea of quantum field theory is that matter and forces are described by *fields* filling all of spacetime. There are equations of motion for these fields, which can have wavelike solutions. In the quantum theory, waves of a given frequency ν cannot have arbitrary energy, but instead come as discrete *quanta*, with energy $E = h\nu$, where h is Planck's constant. The canonical example of this is the electromagnetic field: this is described by Maxwell's equations, whose classical wavelike solutions constitute the electromagnetic spectrum. The quanta of this field are called *photons*, which are often referred to as the particles that carry the electromagnetic force. It is important to bear in mind, however, that the particles of quantum field theory are not quite the same as particles in Newtonian physics: QFT contains both wavelike and particle behaviour, such that which aspect appears in any given experiment depends on what is being measured. This is the famous property of *wave-particle duality*.

Similarly, matter is also described by fields. The electron, for example, is described by a spinor field $\Psi(x^\mu)$, where $x^\mu = (t, \vec{x})$ denotes an arbitrary point in spacetime¹. If you have not seen spinors before, this is simply a mathematical object with four components, such that it has enough degrees of freedom to represent the two possible spin components of the electron (which is a fermion), or its antiparticle (the positron). Furthermore, the field is complex-valued. The equation of motion for this field is the famous *Dirac equation* [6], and any measured quantities involve combinations of the field Ψ and its complex conjugate, such that all observable properties of electrons are real numbers, as they should be. In turn, this means that the equations of the electron are invariant under the transformation

$$\Psi(x^\mu) \rightarrow e^{i\alpha} \Psi(x^\mu) \quad (1)$$

for some constant parameter α , as the factor $e^{i\alpha}$ will be cancelled out by a factor $e^{-i\alpha}$ in the complex conjugate field. We can visualise this transformation as follows. If the electron field is complex-valued, it has a magnitude and a phase at all points in spacetime. At each point, we can represent the phase by an arrow of unit magnitude, pointing in some direction on the unit circle. The transformation of eq. (1) then corresponds to rotating all arrows by the same amount simultaneously at all spacetime points, as shown in figure 2(a), in which the red arrows are obtained by rotating all the black arrows by $\pi/2$, or 90° . This is called a *global gauge transformation* and it is clear that the theory must be invariant under such operations: we can always choose to redefine what we mean by the zero of phase, so that only *differences* in phase are physically meaningful. A global gauge transformation amounts to such a redefinition of the zero of phase.

In fact, QED turns out to have a much more remarkable symmetry than this. The equations of the theory are invariant under a generalisation of eq. (1), in which the phase factor α itself depends on spacetime position:

$$\Psi(x^\mu) \rightarrow e^{i\alpha(x^\mu)} \Psi(x^\mu). \quad (2)$$

In the geometric picture discussed above, eq. (2) corresponds to rotating the arrows describing the phase of the field by different amounts at different spacetime points, as depicted in figure 2(b). This is known as a *local gauge transformation*, and it is not

¹Throughout this review, we work in *natural units* such that $\hbar = c = 1$, where $\hbar = h/2\pi$, and c is the speed of light.

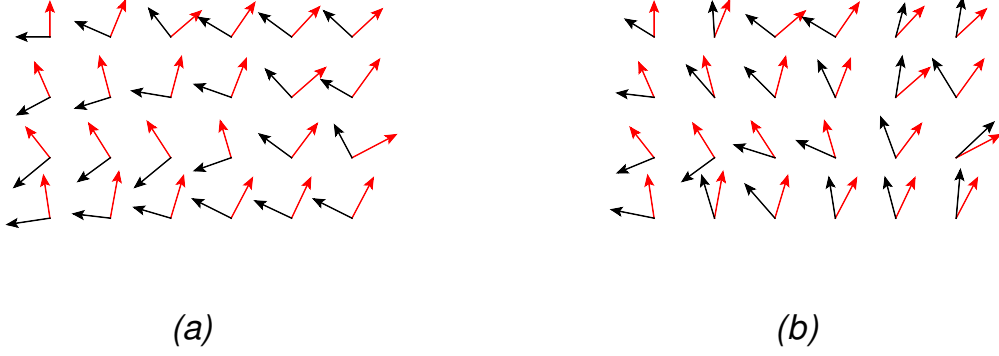


Figure 2. Schematic representation of (a) a global gauge transformation, corresponding to simultaneously shifting the phase of a field at all spacetime points by the same amount (here $\pi/2$, or 90°); (b) a local gauge transformation, in which the phase is shifted by different amounts at each point.

at all obvious a priori that this should be a symmetry of nature. Indeed, local gauge transformations are infinitely richer than global ones, given that there is a separate phase factor for every point in spacetime. Curiously, in order to make the equations describing the electron field invariant under local gauge transformations, one must introduce a 4-vector valued field $A^\mu(x^\mu)$, that couples to the electron in a prescribed way. This field turns out to correspond exactly to the known 4-vector potential in electromagnetism, whose components are

$$A^\mu = (\phi, \vec{A}), \quad (3)$$

with ϕ and \vec{A} the electrostatic and magnetic vector potential respectively. This is a remarkable result: requiring an abstract mathematical symmetry of the electron equations (local gauge invariance) necessarily predicts the existence of electromagnetism, whose equations were first arrived at after hundreds of years of detailed experiments, involving a plethora of different phenomena!

The above discussion immediately begs the question of whether similar abstract symmetries can be used to explain the other forces in the SM, and indeed the answer to this question is yes. Before seeing how this works, let us look at the transformation of eq. (2) more formally. Symmetries are described by the branch of mathematics known as *group theory*. Because the transformations of eq. (2) involve a continuous parameter (α), they form a continuous group, also known as a *Lie group*. The particular group in question is usually denoted $U(1)$, which stands for the set of unitary 1×1 matrices i.e. numbers U satisfying

$$U^\dagger U = U U^\dagger = 1. \quad (4)$$

This already hints at how one can generalise the above gauge symmetry to describe other forces in nature: one can look for invariance under abstract phase transformations corresponding to more complicated Lie groups. The group $U(1)$ is especially simple, given that the transformations commute with each other e.g.

$$e^{i\alpha_1} e^{i\alpha_2} = e^{i\alpha_2} e^{i\alpha_1}. \quad (5)$$

Groups where the elements commute are called *abelian*. The groups describing the

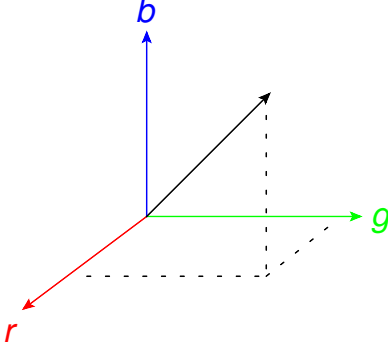


Figure 3. The quark field can be thought of as carrying different components, one for each colour. This leads to an abstract vector space at each point in spacetime, where the arrow specifies how much of each colour is present.

other forces in the SM do not have this property: they are *non-abelian*.

2.2. *Non-abelian gauge theories*

We have seen that abelian gauge invariance can give rise to theories like electromagnetism, in which matter fields interact with a photon field. A famous theorem due to Emmy Noether [7] implies that whenever a physical theory has a symmetry, there must be a conserved quantity. The property of electrons that is conserved, due to local gauge invariance, turns out to be the familiar electric charge e . This charge comes in two types, which we conventionally call *positive* and *negative*. The other forces in the SM also have gauge symmetries associated with them, such that the various matter particles have charges corresponding to each force. These should not be confused with electric charge, and indeed they can have different properties. In particular, it is not true that the different types of charge have only two types.

As an example, let us consider the strong force, felt by quarks (constituents of the proton and related particles). These feel a charge under the strong force, which to avoid confusion with the electric charge is given the name *colour*. Unlike electric charge, colour charge can come in three different types, which are conventionally labelled *red*, *green* and *blue*, or (r, g, b) for short ². It should be stressed that these are merely labels - they have nothing to do with visual colours - and that any other names would have sufficed. Like electrons in QED, quarks are described by a field. There will be a separate field for each colour of quark, which we may collect into a vector

$$\psi_i(x^\mu) = (\psi_r(x^\mu), \psi_g(x^\mu), \psi_b(x^\mu)) \quad (6)$$

where $i \in \{r, g, b\}$ is an index labelling the colour. If we want to, we can visualise this as a single field, but with an abstract vector at each point in spacetime, that tells us how much redness, greenness and blueness the field has - see figure 3. This reminds us of the arrows in figure 2, which described the phase of the electron field at each point. Indeed, it turns out that the equations of the quark field are invariant under a local gauge transformation corresponding to rotating the arrow of figure 3 by arbitrary amounts separately at every point in spacetime. Physically, this corresponds

²Anti-quarks have charges $(\bar{r}, \bar{g}, \bar{b})$ (i.e. *anti-red*, *anti-blue* etc.). We can think of these as being simply related to the three quark charges, or instead count six total types of charge.

to a local redefinition of what we mean by redness, greenness and blueness. As for QED, requiring this invariance requires introducing an additional vector field. This additional field describes the *gluon*, namely the particle that carries the strong force.

Let us surmise the Lie group that corresponds to these gauge transformations. Firstly, moving the arrow in figure 3 translates to some transformation

$$\psi_i \rightarrow \sum_j U_{ij} \psi_j. \quad (7)$$

As for QED, the equations for the quark field depend only on combinations of ψ_i with its complex conjugate, so that all measured quantities are real. Requiring such combinations to be invariant means that the matrix U_{ij} must be *unitary*. This is also consistent with conservation of colour charge, given that a unitary matrix will not change the length of the arrow. We must also rule out reflections (corresponding to inversions of one or more axes), implying that the matrix U_{ij} must have unit determinant. Thus, U_{ij} is a so-called *special unitary* matrix of dimension three, and the group of such transformations is called $SU(3)$. Such a matrix has 8 degrees of freedom, so that demanding local gauge invariance under all possible $SU(3)$ transformations implies that the gluon field must have 8 components. This corresponds to the fact that the gluons themselves carry colour. If, for example, a red quark changes into a green one by emitting a gluon, the latter must supply an outgoing red charge and an incoming green one so that the total colour charge is conserved. There are eight independent combinations of colour charge that a gluon can carry, so that we usually write the gluon field as $A^{\mu a}(x^\mu)$, where μ is the spacetime index, and $a \in \{1, \dots, 8\}$ is the colour index.

A similar mechanism to that explained here can be used to describe all forces in the SM. For the purposes of this review, we can consider a more abstract theory, in which we get rid of the quarks altogether, and consider the gluon field by itself. We may also consider other gauge groups, other than $SU(3)$. Such a theory is more commonly called *Yang-Mills theory*, and it is this that enters the simplest form of the double copy.

3. Scattering amplitudes

A particularly relevant question to ask in a given quantum field theory is what happens when particles interact each other. This is especially topical given that our main way of testing potential new physics theories is to use *particle accelerators*, which collide beams of particles together, before measuring the resulting debris. A convenient way to depict particle interactions in a given theory is to use *Feynman diagrams*. Roughly speaking, we can think of these as space-time diagrams showing the history of a scattering event, where different types of line represent various types of particle³. Some simple examples are shown in figure 4, which shows the different possibilities for two incoming gluons to produce two outgoing gluons. Each curly line represents a gluon, which carries an index a corresponding to the colour index in the gauge field A_μ^a . Furthermore, each incoming or outgoing gluon will have a 4-momentum

$$p_i^\mu = (E_i, \vec{p}_i), \quad (8)$$

³Feynman diagrams actually represent a Lorentz-invariant sum of possible scattering histories, to be consistent with special relativity, so should not be thought of too literally!

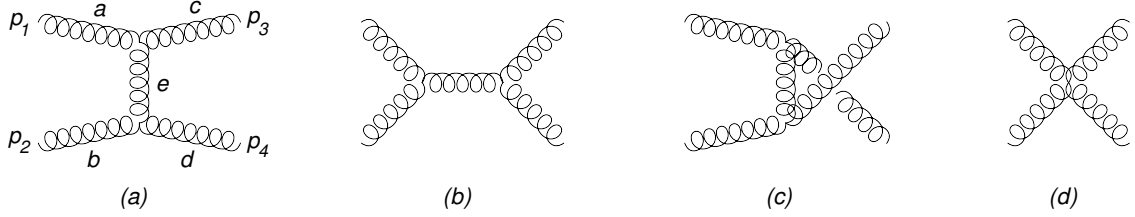


Figure 4. Feynman diagrams for two gluons (shown as curly lines) scattering to two gluons: (a) the first gluon emits a gluon that is absorbed by the second, or vice versa; (b) two gluons annihilate to make a single gluon, which splits again into two; (c) similar to process (a), but with the final state momenta interchanged; (d) two gluons annihilate and immediately produce two gluons with different momenta. In diagram (a), we have labelled the colour index of each gluon, and the 4-momenta of the initial and final state gluons.

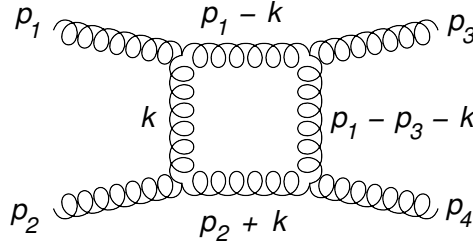


Figure 5. A loop diagram, in which four external particles exchange a pair of virtual gluons. Once the momentum k is specified, this fixes all other momenta as shown.

where E_i and \vec{p}_i are the (relativistic) energy and 3-momentum respectively. The rules of quantum field theory tell us that, for a given number of incoming and outgoing particles, we must draw all distinct connected diagrams containing a given set of vertices, where the latter are determined by the theory. In Yang-Mills (pure gluon) theory, there are vertices connecting either three or four gluons at a single point, leading to the four diagrams of figure 4. Note that for graphs (a)–(c), there is an internal line in addition to the four external gluon lines. This represents a so-called *virtual particle*, that is not observed directly, but exchanged between incoming or outgoing particles. Four-momentum is conserved at all vertices. For example, in diagram (a) we can associate the 4-momentum

$$q^\mu = (p_1 - p_3)^\mu \quad (9)$$

with the internal line, where the minus sign on the right-hand side arises from the fact that p_1 is flowing into the vertex, but p_3 is flowing out (we have also chosen q^μ to flow outwards). The conservation of 4-momentum at all vertices of a Feynman diagram is reminiscent of Kirchhoff's current law for electrical currents, and can be applied just as straightforwardly.

The diagrams of figure 4 are the simplest ones we can draw that connect all four external particles. For a given number of external lines, we can also add more vertices by including more virtual particles. An example is shown in figure 5, in which the two incoming gluons exchange a pair of virtual gluons. If loops are present, the external momenta are no longer sufficient to determine the momenta of all internal lines. We must fix a momentum k (we can choose which one), after which all the other momenta are determined, as shown in the figure. You can easily convince yourself that if L loops are present, one needs to specify the momenta of L internal lines in this way.

These momenta are called *loop momenta*, and diagrams with no loops (such as those of figure 4) are called *tree-level diagrams*, as the corresponding graphs constitute trees in graph theory parlance.

Feynman diagrams gives us a convenient way to visualise how particle interactions happen in quantum field theory. However, they are much more than this. It turns out that there is a precise set of so-called *Feynman rules*, that convert each diagram into a mathematical contribution to a complex number called the *scattering amplitude* \mathcal{A} , where $|\mathcal{A}|^2$ is related to the probability for the interaction to occur. In order to state the rules, one must break the gauge invariance of the theory. This is called *choosing a gauge*, and different Feynman rules exist for different gauges. However, upon adding together the contributions from all possible diagrams, the resulting scattering amplitude becomes gauge-invariant, as it should be given that this is a symmetry of the theory. A common gauge choice for Yang-Mills theory is the *Feynman gauge*, which says that each internal line of 4-momentum q^μ is associated with a factor

$$D_{\mu\nu}^{ab} = -\frac{i\delta^{ab}\eta_{\mu\nu}}{q^2}, \quad (10)$$

where (a, b) and (μ, ν) are respectively the colour and Lorentz indices associated with each end of the line. Furthermore, δ^{ab} and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ are the Kronecker delta symbol and the metric tensor for Minkowski space. We see that every internal line involves an inverse power of the squared 4-momentum flowing through the line.

Each three-gluon vertex is associated with a factor

$$V_{\mu_1\nu_1\nu_2}^{a_1a_2a_3}(p_1, p_2, p_3) = -gf^{a_1a_2a_3} [(p_1 - p_2)_{\mu_3}\eta_{\mu_1\mu_2} + (p_2 - p_3)_{\mu_1}\eta_{\mu_2\mu_3} + (p_3 - p_1)_{\mu_2}\eta_{\mu_3\mu_1}]. \quad (11)$$

Here g is a *coupling constant* representing the strength of the force between interacting gluons (in the full theory of QCD, it also describes the interaction strength between gluons and quarks). Each gluon of momentum p_i entering the vertex has an associated Lorentz index μ_i and colour index a_i . Furthermore, $f^{a_1a_2a_3}$ is a known number which depends on the colour charges of the three gluons⁴. It is antisymmetric under the interchange of any two indices:

$$f^{abc} = f^{bca} = f^{cab} = -f^{acb} = -f^{cba} = -f^{bac}, \quad (12)$$

and satisfies the *Jacobi identity*

$$f^{abe}f^{ecd} + f^{ace}f^{edb} + f^{ade}f^{ebc} = 0, \quad (13)$$

where summation over repeated indices is implied. With each incoming gluon, we must associate a 4-vector $\epsilon_\mu(p_i)$ (or its complex conjugate if the gluon is outgoing) describing its polarisation. The factors associated with each internal line, vertex and external particle are then simply multiplied together for each Feynman diagram, with sums over all repeated Lorentz and colour indices. There is also a rule for the four gluon vertex appearing in figure 4(d). However, it turns out that one can always choose

⁴More specifically, for any given gauge group, the quantities $\{f^{a_1a_2a_3}\}$ are the structure constants of the associated Lie algebra.

to rewrite such diagrams to involve sums of products of three-gluon vertices, so we will not need the four-vertex for the following discussion.

For loop diagrams, an additional rule is needed. Given that each loop has an associated loop momentum k^μ , we need to sum over all the possible values that this momentum can have. This takes the form of an integral over all 4 components. Finally, for all diagrams, one must divide by the number of symmetry transformations S_i that leave the graph invariant. From this and the above rules (including the conservation of 4-momentum at all vertices), we can write a very general schematic formula for any scattering amplitude in Yang-Mills theory, involving an arbitrary number m of external particles, and involving L loops ⁵:

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_i \int \prod_{l=1}^L \frac{d^D k_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \quad (14)$$

The sum is over all cubic graphs i (where it is assumed all four-vertices have been eliminated). The overall power of the coupling constant g is equal to the number of cubic vertices. For each graph, there is an integral over the loop momenta $\{k_l\}$ (L of them), and a denominator involving the product of all momenta associated with internal lines. The numerator for each graph will contain two contributions. Firstly, there is a part depending on the colour charges of the gluons. From eq. (11), this is simply given by the product of the factors $f^{a_1 a_2 a_3}$ at each vertex. For example, the colour factors of the diagrams in figure 4(a)–(c) are given by ⁶

$$c_a = f^{ace} f^{edb}, \quad c_b = f^{aeb} f^{ecd}, \quad c_c = f^{ade} f^{ecb}. \quad (15)$$

Secondly, there is a kinematic factor n_i , depending on the external and loop momenta, as well as the polarisation vectors for the external particles. Note that individual terms in eq. (14) do not necessarily come from individual Feynman diagrams: for example, the four-gluon vertex in the latter is rewritten so as to contribute to multiple terms in the sum. Calculating the amplitude then amounts to determining the kinematic numerators $\{n_i\}$ for each graph. They are not unique - their explicit form will depend on the choice of gauge, as well as other freedoms that we will see later. However, the scattering amplitude is unique for a given theory, so that any ambiguities must cancel in the sum in eq. (14).

We have so far described scattering amplitudes in a non-abelian gauge theory. However, we can also talk about them in quantum gravity, which means identifying a “graviton field” that is analogous to the field A_μ^a describing the gluon in a gauge theory, and which carries the force of gravity. We have already seen above that General Relativity associates gravity with the structure of space-time itself. We can represent this mathematically as follows. Consider two events in spacetime, separated by an infinitesimal distance

$$dx^\mu = (dt, d\vec{x}), \quad (16)$$

where dt and $d\vec{x}$ are the temporal and (vector) spatial separation respectively. In the

⁵To be fully general, we present this formula in D spacetime dimensions, where our apparent spacetime has $D = 4$.

⁶To reproduce these expressions, one must assign the colour indices clockwise at each vertex. We have labelled colour indices for the external and internal gluon for all graphs as in figure 4(a).

absence of gravity, we can associate a Lorentz-invariant spacetime distance

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dt)^2 - (d\vec{x}) \cdot (d\vec{x}), \quad (17)$$

where repeated indices are summed over, and $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the metric tensor of flat space. When gravity is present, this equation becomes modified to

$$ds^2 = g_{\mu\nu}(t, \vec{x}) dx^\mu dx^\nu. \quad (18)$$

That is, the metric tensor becomes a function of spacetime position, so that the distance measure changes as we move throughout the space. The space is thus warped, and the form that $g_{\mu\nu}$ takes is dictated by the distribution of mass and energy throughout the spacetime. The precise relationship is given by *Einstein's field equations*, which take the form of a differential equation relating second derivatives of the metric tensor (sensitive to the curvature of spacetime) to an *energy-momentum tensor* describing the matter distribution.

For weak gravitational fields, we know that the metric tensor must reduce to its flat space counterpart. This allows to write

$$g_{\mu\nu}(t, \vec{x}) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(t, \vec{x}) \quad (19)$$

for some number κ , such that $h^{\mu\nu}$ is a field representing the presence of gravity i.e. the deviation from flat space. Substituting this into the Einstein equations, we may fix κ by requiring consistency with Newton's theory of gravitation for sufficiently weak fields (such that h^{00} represents the Newtonian potential), which leads to $\kappa = \sqrt{32\pi G_N}$, where G_N is Newton's constant. The number κ thus represents the strength of the gravitational force i.e. it is the analogue of the coupling constant g in a non-abelian gauge theory.

Substituting eq. (19) into the equations of General Relativity leads to an equation for the field $h_{\mu\nu}(x)$, whose solutions include the *gravitational waves* recently discovered by LIGO [1]. The equation is highly nonlinear, where the nonlinear terms correspond to interactions between the field $h_{\mu\nu}$ and itself, in much the same way that the gluon can interact with itself in gauge theory. Similarly to the latter case, we can attempt to quantise the theory, such that gravitational waves come in discrete packets called *gravitons*. We can then consider scattering amplitudes for gravitons, which will be given by similar Feynman diagrams to those we have already seen for gluons. However, the Feynman rules will now be different, as they will be governed by the equations of General Relativity rather than Yang-Mills theory.

This procedure was first carried out in earnest in the 1960s [8–11]. As for non-abelian gauge theories, the Feynman rules are not unique: given that the graviton $h_{\mu\nu}$ has two free spacetime indices, the rules will depend on which coordinate system is chosen, such that this ambiguity cancels out in the final scattering amplitude. The fact that final results must be independent of the coordinate system is known as *diffeomorphism invariance*, and it is possible to understand this as a type of gauge symmetry acting on the graviton field $h_{\mu\nu}$. Fixing a coordinate system for the Feynman rules is then referred to as *choosing a gauge*, by analogy with the Yang-Mills case. In the commonly chosen *De Donder gauge*, the three-graviton vertex has well over a hundred individual terms, in stark contrast to the three-gluon vertex in Yang-Mills theory, which has only six terms, as can be seen in eq. (11). Vertices involving four or more gravitons are even more complicated, such that calculations in quantum gravity are stupendously

unwieldy, rapidly becoming intractable as either the number of loops in Feynman diagrams, or the number of external particles, is increased. The use of powerful computers can help, but available computing power still limits which calculations are possible. As discussed above, results obtained in pure General Relativity indicate that this theory contains divergences at higher loop orders which cannot be removed by simply redefining the parameters of the theory (i.e. the theory is non-renormalisable). It is possible that adding extra matter content or symmetries to the theory may help, but investigation of this question is clearly hampered by the difficult nature of amplitude calculations.

Like Yang-Mills theory, individual Feynman diagrams in gravity are not gauge-invariant by themselves, such that the symmetry is only restored upon combining all diagrams to form the full amplitude. Remarkably, the final results are often much simpler (exceedingly so in gravity) than the individual diagrams would suggest. Literally thousands of terms *cancel* in the sum over diagrams, suggesting that much of the complexity in Feynman diagram calculations is associated with gauge-dependent artifacts, that are physically irrelevant, and thus ultimately absent in the final results.

The above considerations have led to the development of alternative methods for calculating amplitudes, that don't necessarily use Feynman diagrams as a starting point (see e.g. refs. [12,13] for pedagogical reviews). The picture that is emerging is that gravity amplitudes, if thought about in the right way, possess a simplicity and elegance that is almost entirely hidden using traditional calculational methods. Furthermore, there are deep and profound similarities between gravity amplitudes and their counterparts in non-abelian gauge theories, that may have far-reaching consequences.

4. BCJ duality and the double copy

In eq. (14) we saw that a scattering amplitude in a non-abelian gauge theory can be written in a very general form, as a sum over cubic graphs. Each of these has a colour factor obtained by dressing each vertex with a factor f^{abc} , where $\{a, b, c\}$ denote the colour indices of the gluons entering the vertex. Furthermore, these factors satisfy the Jacobi identity of eq. (13), which allows us to relate the colour factors of different Feynman diagrams. As an example, the results of eq. (12, 13, 15) imply that the colour factors of the diagrams in figure 4(a)–(c) are related by

$$c_a - c_b - c_c = 0. \quad (20)$$

Similar relations apply between more complicated diagrams, involving more external particles, or loops. An example is shown in figure 6, which shows three diagrams arising at one-loop order. Each diagram consists of a part X common to all three, which will give rise to the same contribution c_X to each colour factor. There is then a part which looks like one of the diagrams of figure 4(a)–(c). We can then write the colour factor of each diagram as

$$c_A = c_X c_a, \quad c_B = c_X c_b, \quad c_C = c_X c_c, \quad (21)$$

such that eq. (20) implies

$$c_A - c_B - c_C = 0. \quad (22)$$

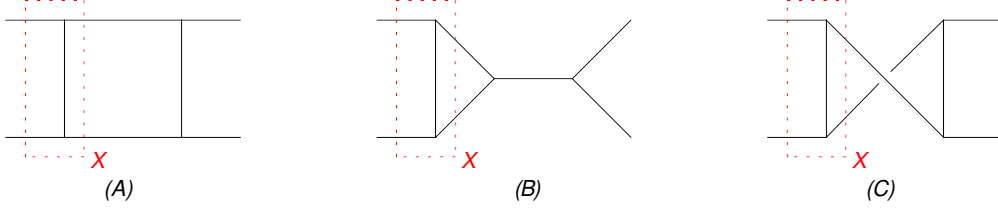


Figure 6. Three Feynman diagrams occurring in Yang-Mills theory at one-loop order, where gluons have been shown as solid lines for brevity. The highlighted region X is the same in each diagram, and thus contributes a common colour factor.

Generalising this, we find that at any number of loops L , and for any number of external particles m , we can always arrange diagrams into overlapping sets of three, each obeying a *Jacobi relation* such as those of eqs. (20, 22). These relations are a direct consequence of the non-abelian gauge symmetry of the theory, which gives rise to the properties of eq. (12, 13).

In 2008, a remarkable new property of tree-level ($L = 0$) gauge theory amplitudes was discovered [5]: whenever a given set of three diagrams $\{i, j, k\}$ obeys a colour Jacobi relation, the corresponding kinematic numerators (as functions of momenta) can be chosen to obey a similar relation:

$$c_i \pm c_j \pm c_k = 0 \quad \Rightarrow \quad n_i \pm n_j \pm n_k = 0. \quad (23)$$

Furthermore, the numerators $\{n_i\}$ can be chosen to obey similar antisymmetry conditions to eq. (12), but where kinematic rather than colour degrees of freedom are interchanged. This property is known as *BCJ duality*, and had not been previously appreciated, for understandable reasons. Recall that the kinematic numerators are not unique, but depend e.g. on the gauge chosen for calculating Feynman diagrams. In arbitrary gauges (including those most commonly used for calculations), the numerators *will not* obey eq. (23). However, it is always possible to shift the numerators according to

$$n_i \rightarrow n_i + \Delta_i, \quad (24)$$

where Δ_i is a function of momenta, such that BCJ duality is satisfied. This is called a *generalised gauge transformation*, as it combines a gauge transformation with other possible operations, such as a redefinition of the gluon field. Substituting eq. (24) into eq. (14), a generalised gauge transformation is permitted provided the shifts $\{\Delta_i\}$ satisfy

$$\sum_i \frac{1}{S_i} \frac{\Delta_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0, \quad (25)$$

which restricts the possible form of each Δ_i , but still allows a great deal of freedom.

BCJ duality is proven to be possible at tree-level and has been conjectured to hold at all loop orders, due to highly nontrivial evidence. For example, we have here described the simplest theory in which BCJ duality applies, namely pure Yang-Mills theory. However, supersymmetric generalisations of this theory exist, in which additional fermionic and scalar matter is added, whose couplings are such that the equations of the theory are invariant under interchanging bosonic and fermionic degrees of free-

dom. Loop calculations become easier in such theories, and in a particular example ($\mathcal{N} = 4$ Super-Yang-Mills theory), BCJ duality has been observed up to four-loop order [3,14]. It has also been seen at one and two-loop order even if supersymmetry is not present [15], and thus BCJ duality in some form is certainly a genuine quantum property of nature.

We saw that the colour Jacobi relations are a manifestation of non-abelian gauge symmetry. The fact that the kinematic numerators satisfy similar relations implies that there must also be a (different) symmetry underlying them. What this symmetry might be remains unknown. It is also unclear whether such a symmetry underlies the full equations defining the theory, or is merely an accidental property of scattering amplitudes. Intriguing hints, however, come from *self-dual Yang-Mills theory*, which is obtained from Yang-Mills theory by keeping only a single, particular polarisation state of the gluon. In this case, the kinematic symmetry has been associated with a certain area-preserving diffeomorphism group [16]. If more general results for full Yang-Mills and related theories can be elucidated, it would constitute a hitherto undiscovered structure in non-abelian gauge theory, that may provide deep insights into why such theories are so fundamental in nature. Even without this, imposing BCJ duality may be a useful tool for calculating new scattering amplitudes (see e.g. ref. [17]).

Once the numerators in eq. (14) have been chosen to make BCJ duality manifest, a second remarkable conjecture links gauge theories with gravity, as follows. First, one may replace the gauge theory coupling g with its gravitational counterpart, according to

$$g \rightarrow \frac{\kappa}{2}. \quad (26)$$

Next, one can strip off the colour factors c_i for each cubic graph, and replace them with a second set $\{\tilde{n}_i\}$ of kinematic numerators. The tilde on these quantities indicates that they do not necessarily come from the same gauge theory as the $\{n_i\}$. Upon making these modifications, one arrives at the quantity ⁷

$$\mathcal{M}_m^{(L)} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_i \int \prod_{l=1}^L \frac{d^D k_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}. \quad (27)$$

The *double copy* conjecture ⁸ [3,4] – which is not at all obvious a priori – states that this formula corresponds to a gravitational scattering amplitude with m external particles and L loops. We will address how we know this is a gravity amplitude in more detail below. Even without knowing any physics at all though, it is clear from sight alone that eq. (27) is extremely similar to its gauge theory counterpart, eq. (14)! The name “double copy” refers to the fact that two gauge theory numerators occur in the gravity formula. This is sometimes also expressed by saying that gravity is the “square” of gauge theory, although we do not mean that one literally squares the gauge theory amplitude to obtain the gravity result: each term is copied separately, and denominator factors are left untouched.

It is difficult to exaggerate how remarkable eq. (27) is. The traditional calculation of gravity amplitudes, as discussed above, is extraordinarily complicated, involving

⁷The additional factor of i in eq. (27) relative to eq. (14) is conventional, and will cancel in the squared amplitude.

⁸It can be proven that the double copy holds provided BCJ duality does at higher loop orders, as explained in ref. [4].

a forbidding quagmire of algebraic complexity. Eq. (27), however, suggests that the basic structure of gravity amplitudes is essentially identical to those of gauge theory, despite the fact that the Feynman rules are orders of magnitude more complicated!

We have not yet stated which gravity theory the amplitude of eq. (27) resides in, and indeed the answer depends on which two gauge theories are chosen for the two sets of kinematic numerators. The simplest form of the double copy consists of pure Yang-Mills theory being copied with itself. This leads to General Relativity coupled to an additional scalar particle (known as the *dilaton*), and an antisymmetric tensor field. In four dimensions, the extra particles constitute two degrees of freedom. Combined with the two polarisation states of the graviton, this gives four degrees of freedom in total. This matches up with the fact that we have “copied” two gauge theories with 2 degrees of freedom each (the two polarisation states of the gluon). More complicated examples of the double copy involve e.g. supersymmetric generalisations of Yang-Mills theory. If we have $\mathcal{N} = N$ supersymmetries (relations between bosons and fermions) in the first gauge theory, and $\mathcal{N} = M$ in the second, the number of supersymmetries in the resulting gravity theory is $\mathcal{N} = (N + M)$. Again, the number of degrees of freedom (i.e. the number of individual polarisation states) on both sides of the correspondence have to match up.

How do we know that eq. (27) corresponds to a gravity amplitude (possibly with additional matter)? First of all, we can compare known gravity amplitudes with the result of eq. (27), and verify the agreement directly. Alternative methods exist for determining whether a previously unknown amplitude is indeed physically correct (see e.g. [12] for a review). Further evidence for the correctness of the double copy conjecture comes from examining particle scattering in certain special energy regimes. In the limits when particles have a very high centre of mass energy [18–23], or when very fast-moving particles exchange very low energy gravitons or gluons [24–30], the double copy can be verified for any number of loops L . Beyond this, the lack of a formal proof of the double copy is related to the lack of a formal proof of BCJ duality beyond tree level.

The double copy consists of stripping off the colour factors from a gauge theory amplitude, and replacing them with extra kinematic factors. We could also do the opposite, namely replacing the kinematic factors $\{n_i\}$ with a second set of colour factors $\{\tilde{c}_i\}$. This is called the *zeroth copy*, and yields the quantity

$$\mathcal{T}_m^{(L)} = i^L y^{m-2+2L} \sum_i \int \prod_{l=1}^L \frac{d^D k_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i \tilde{c}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \quad (28)$$

where we have also replaced the coupling $g \rightarrow y$. We can then ask if eq. (28) corresponds to a scattering amplitude in some theory, and the answer is indeed yes. The relevant theory is called *biadjoint scalar theory*, and consists of a scalar field $\Phi^{aa'}$ that carries two different types of colour charge, labelled by a and a' . The field interacts with itself, where y describes the strength of this interaction. Although this is not a physical theory itself, there is mounting evidence that dynamics in this theory is inherited, via the double copy, by gauge theory and gravity. The network of theories which are related by similar relationships continues to widen [31–34].

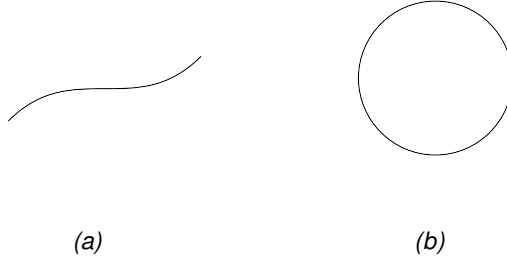


Figure 7. (a) An open string; (b) a closed string.

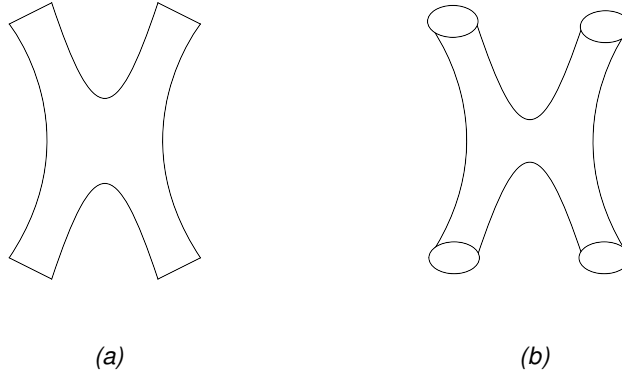


Figure 8. A string moving through spacetime traces out a worldsheet. These combine into Feynman diagrams, shown here for (a) four open strings interacting; (b) four closed strings.

5. Insights from string theory

For tree-level diagrams (i.e. with no loops), the double copy turns out to be equivalent to a previously discovered relationship between scattering amplitudes in gauge and gravity theories, which can be derived from *string theory*. The latter is a candidate for a possible *theory of everything*, in which all the Standard Model forces and gravity can be combined into a single consistent theoretical description. The basic idea of string theory is that what we see as particles at currently accessible energies are in fact string-like at very small distance scales (from the uncertainty principle, this is equivalent to very high energies)⁹. There are two types of string: *open* and *closed*, as shown in figure 7. Strings can interact with each other, such that we can draw their interactions using Feynman diagrams. However, whereas a particle traces out a *worldline* as it moves through spacetime, a string traces out a two-dimensional *worldsheet*. Examples for open and closed strings are shown in figure 8. It is known how to calculate scattering amplitudes for strings, and in the absence of loops, the *KLT relations* state that amplitudes for closed strings can be written as sums of products of amplitudes for open strings [35]. The low energy limit of string theory is a quantum field theory, where open strings give rise to particles like the gluon, and closed strings correspond to gravitons. Thus, the KLT relations in string theory imply a set of double copy relations between scattering amplitudes in non-abelian gauge theories and gravity. Note in this argument that it is completely irrelevant whether or not string theory is a genuine physical theory of nature. Here, it merely acts as a

⁹String theory also necessarily contains higher-dimensional objects called *branes*, that we will not need to discuss here.

mathematical bridge between two types of theory that definitely do exist in the real world.

Unfortunately, the string theory argument is not straightforward to generalise to diagrams with loops, so that whether or not it provides a full justification for the double copy remains unclear. Recent developments, however, may shed light on this issue. Firstly, it is possible to write field theory scattering amplitudes in a form that looks very like a string theory, using the so-called *Cachazo-He-Yuan (CHY) equations* [36–38]. Indeed, the same formula has been shown to be obtainable from a string theory living in *ambitwistor space* [39–41], which can itself be obtained as a limit of string theory in conventional spacetime [42]. Loop corrections can be calculated in this theory [43,44], and the CHY equations are written in such a way that the double copy between gauge and gravity theories becomes particularly clear. Thus, this may indeed provide a framework for exploring the underlying origin of the double copy. In a different approach, ref. [45] (following earlier related work that predates the double copy [46]) rewrites the equations defining General Relativity in a new form, which aims to make its double copy structure more manifest.

6. The classical double copy

Up to now, our discussion of the double copy has been restricted solely to scattering amplitudes, which are approximate solutions of the equations of quantum field theory (i.e. they correspond to an expansion in the coupling constant). It is natural to ponder whether the double copy goes much deeper than this, in which case it constitutes a very profound relationship between gauge and gravity theories, providing a common way to think about two previously orthogonal physical frameworks. One way to examine this is to look at *exact* solutions of gauge and gravity theories. This is a two-step procedure: (i) one must find a way to associate a gauge field A_μ^a with a given exact graviton solution $h_{\mu\nu}$ of general relativity; (ii) one must show that the double copy relationship thus obtained is consistent with the BCJ double copy for amplitudes [3,4].

This idea was first explored in ref. [47], which considered the special class of *Kerr-Schild* metric tensors in General Relativity. These take the form of eq. (19), where the graviton field is given by

$$h_{\mu\nu} = \phi k_\mu k_\nu. \quad (29)$$

Here $\phi \equiv \phi(t, \vec{x})$ is a scalar field, and we see that the graviton consists of an outer product of a 4-vector $k_\mu \equiv k_\mu(t, \vec{x})$ with itself. This vector is not arbitrary, but must satisfy the following two conditions:

$$k^2 = k^\mu k_\mu = 0, \quad k^\mu \partial_\mu k^\nu = 0. \quad (30)$$

Substituting the ansatz of eq. (29) into the Einstein equations results in them becoming *linear*. They are then relatively easy to solve for ϕ and k_μ , where different solutions correspond to different possible spacetimes. Furthermore, the resulting solutions are *exact*, with no higher order corrections in κ .

The simple form of eq. (29) suggests how one can form a gauge field corresponding to the gravity solution: given that the gauge field should contain one spacetime index rather than two, we can strip off one copy of the Kerr-Schild vector k_ν , and replace it with an arbitrary constant vector in the abstract colour space of the gauge theory.

The resulting gauge field then has the form

$$A_\mu^a = c^a \phi k_\mu, \quad (31)$$

and ref. [47] proved that a gauge field thus obtained from any stationary Kerr-Schild graviton (i.e. with no explicit time dependence) automatically satisfies the equations of Yang-Mills theory. These equations also become linear, so that the solution is exact, with no higher order corrections. This fulfills the first condition above, of finding an exact map between solutions of gauge theory and gravity. However, the procedure appears ambiguous: why should one strip off the vector k_μ , and not, say, absorb some (or all) of the field ϕ whilst doing so? In other words, what fixes the double copy procedure between the two theories? The answer can be obtained by considering the *zeroth copy*. Repeating the above procedure, one obtains a biadjoint scalar field

$$\Phi^{aa'} = c^a \tilde{c}^{a'} \phi, \quad (32)$$

where we have replaced the Kerr-Schild vector in the gauge theory solution by a second colour vector. The resulting field turns out to be an exact solution of the theory whose amplitudes are given by eq. (28). Furthermore, the fact that one does not modify the scalar field ϕ when double copying exact solutions from gauge theory to gravity can be related directly to the fact that one does not modify denominator factors when double copying the amplitudes of eqs. (14, 27, 28). This fulfills the second condition above, of showing that the double copy for exact solutions is consistent with the similar procedure for amplitudes.

Although the class of stationary Kerr-Schild solutions is rather special, it is infinitely large, and includes well-known exact solutions of General Relativity (albeit some that are more conventionally presented in alternative coordinate systems). One example is the Schwarzschild black hole, a static, non-rotating black hole that would result, for example, from a completely symmetric collapsing star. The metric for this solution can be written in the Kerr-Schild form of eq. (29), where

$$\phi = \frac{M}{4\pi r}, \quad k^\mu = (1, \hat{e}_r), \quad (33)$$

where r is the radial distance from the origin (where the centre of the black hole is), and \hat{e}_r a unit 3-vector in the radial direction. The resulting gauge field, after making a gauge transformation and the coupling replacement of eq. (26), is found to be ¹⁰

$$A^{\mu a} = \frac{c^a}{4\pi r} (1, \vec{0}). \quad (34)$$

This is the well-known *Coulomb solution*, corresponding to a static point charge at the origin. Thus, the single copy or “square root” of a Schwarzschild black hole (sourced by a point-like mass), is a point-like colour charge! This is conceptually similar to the amplitude story, where colour (charge) information in the gauge theory gets replaced by kinematic (i.e. momentum) factors.

A more complicated black hole is the *Kerr black hole*, corresponding to a rotating disk of mass of finite size. The corresponding gauge theory solution is found, as ex-

¹⁰The mass M in eq. (33) is also replaced with a colour-charge dependent factor that does not appear in eq. (34), as described in ref. [47].

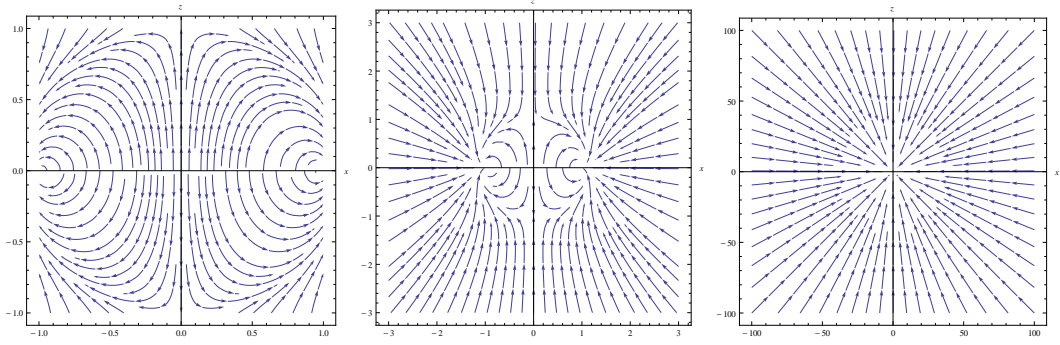


Figure 9. The electric field generated by the single copy of the Kerr black hole, where the charged disk has unit radius and lies on the horizontal axis.

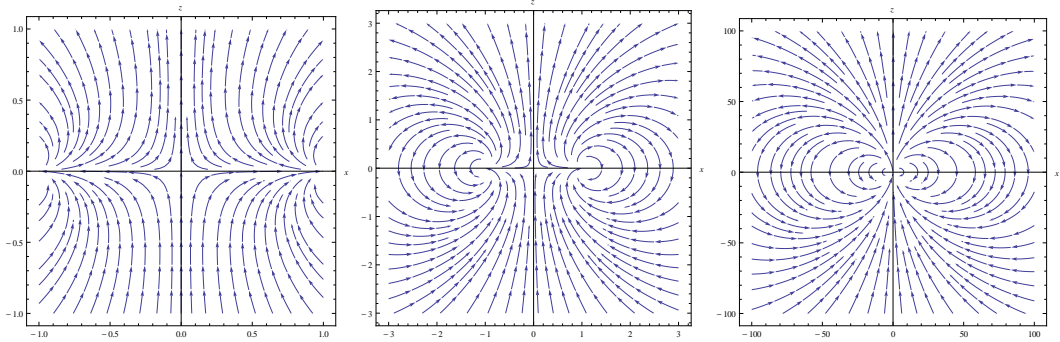


Figure 10. The magnetic field generated by the single copy of the Kerr black hole, where the charged disk has unit radius and lies on the horizontal axis.

pected, to be a rotating disk of charge, whose profile mirrors the mass profile in the gravity theory [47]. One can plot the electric and magnetic fields in the gauge theory¹¹, as we show in figures 9 and 10. Both fields have non-trivial structure at short distance, due to how the charge is distributed on the disk. However, at large distances the electric field looks like a point charge (i.e. points radially outwards) as it should. Furthermore, the magnetic field is dipole-like, which is what one expects for a coil of current. A rotating charge disk indeed looks like a set of current coils, thus the results make sense. Analogues of both the Schwarzschild and Kerr black holes exist in higher dimensions [48,49], and the double copy works there too, which is consistent with the fact that the BCJ copy for amplitudes is independent of the number of spacetime dimensions D in eqs. (14, 27, 28).

Whilst the fields double copy between gauge theory and gravity, the same is not always true for the sources. For example, the energy-momentum tensor that sources the Kerr metric is not a straightforward double copy of the current corresponding to the rotating charge disk in the gauge theory, but instead includes additional radial pressure terms, which are needed to stabilise the mass disk in gravity [47]. The nature of sources has been further examined in ref. [50].

More recent work has tried to extend the double copy for exact solutions beyond the special case of Kerr-Schild. It is possible, for example, to look at *multi-Kerr-Schild* solutions, in which additional terms of the form of eq. (29) are added to the metric,

¹¹Given that the Yang-Mills equations linearise for this solution, they behave just like Maxwell's equations, so that we can talk about electric and magnetic fields in the same way.

involving different scalar fields and 4-vectors. Such an ansatz is no longer guaranteed to linearise the Einstein equations, but a known case where this indeed happens is the *Taub-NUT metric* [51], originally derived (in a different coordinate system) in refs. [52,53]. This solution has a Schwarzschild-like mass at the origin, as well as a so-called *NUT charge* that gives rise to a rotational character of the gravitational field at spatial infinity. The single copy was constructed in ref. [54] and consists of a *dyon* having both electric and magnetic monopole charges, which map to the mass and NUT charge in gravity respectively. Analogies between the Taub-NUT solution and magnetic monopoles, at large radial distances, have been made before. However, the double copy makes this relationship exact, for arbitrary distances.

Another extension away from stationary Kerr-Schild metrics is to consider time dependence. Arguably the simplest time-dependent solutions are plane waves [55,56], which are known to double copy between gauge theories and gravity [56–58], but which can also be cast in Kerr-Schild language [47]. Wave-like solutions have also been recently used to construct a double copy with a curved-space (non-Minkowski) background for the graviton [59]. Another time-dependent system is that of an arbitrarily accelerating particle, considered in ref. [60]. This has a known Kerr-Schild description (see e.g. ref. [61]), but has the unusual feature that radiation appears as a source term in both the Yang-Mills and gravity equations [60]. As explained in the latter reference, this can itself be related to known amplitudes for the emission of gluon and graviton radiation in the low energy limit, strengthening the link between the Kerr-Schild and amplitude double copies yet further.

Kerr-Schild solutions are useful in that they linearise the field equations, and hence are exact solutions. One can also consider classical fields that are not known exactly, but must be constructed order-by-order in the coupling (analogous to how scattering amplitudes are calculated in perturbation theory). Indeed, such fields are routinely calculated in order to compare GR with astrophysical measurements, including those involving gravitational waves. Even in classical GR, these calculations are highly complex, and would be greatly simplified by being able to perform simpler calculations in a gauge theory, before doubly copying them to gravity. This has been investigated recently in refs. [62–64] (see also [65,66]). Further work is needed, however, in order to remove the additional matter particles that appear alongside gravity in the double copy, and also to push the calculations to higher powers of the coupling.

An orthogonal body of work has looked at constructing gravity solutions, with and without supersymmetry, using convolutions of gauge fields [32,33,67,68]. This approach has the advantage of being independent of the gauge chosen on the gauge theory side, but is currently set up only to linear order.

Finally, all of the above examples of the double copy rely on solutions of the linearised field equations, in biadjoint scalar, gauge or gravity theories. If a true double copy relationship exists between these theories, it should be true also for fully *non-linear* solutions, involving e.g. inverse powers of the coupling. Preliminary steps to investigate this have been taken in ref. [69,70], which derived a number of nonlinear classical solutions of biadjoint scalar theory, with the hope of using these as building blocks for such solutions in gauge theory. Indeed, an intriguing relationship was noted between these solutions and *Wu-Yang monopoles* [71] in Yang-Mills theory.

7. Conclusion

The double copy is a new and remarkable relationship between gauge and gravity theories, which underlies the four fundamental forces in nature. Furthermore, more exotic theories (such as biadjoint scalar theory) can be shown to obey double-copy like behaviour, such that the dynamics of both gauge and gravity theories may be a lot simpler than previously thought.

There are a number of both theoretical and practical applications of the double copy. Firstly, it greatly streamlines calculations in both classical and quantum gravity. This has renewed the investigation of whether alternative field theories of gravity - such as $\mathcal{N} = 8$ Supergravity - are well-behaved in the quantum regime [72,73]. The calculation of gravitational observables for use in astrophysics may potentially also be drastically simplified, by essentially replacing GR with double-copied Yang-Mills theory. Such techniques may also prove useful for cosmology. The new way to think about gravity provided by the double copy may give clues about how to unify all the forces in nature, and has also tightened up our understanding of how field and string theories are related, where the latter can be used to link various field theories together.

We conclude by stressing that the double copy is a relatively recent discovery, and that many aspects of this fascinating correspondence remain to be explored. Given the wide range of physical phenomena that are linked by the double copy, it is certainly possible for researchers from a wide range of fields to actively contribute to ongoing efforts in this area!

Author biography



Dr. Chris White is a Reader in Theoretical Physics at the Centre for Research in String Theory, Queen Mary University of London. He completed his PhD on the structure of the proton at high energy at the University of Cambridge in 2006, followed by research appointments at the Dutch National Institute for Nuclear and High Energy Physics (Nikhef) in Amsterdam, and Durham University. He then held a lectureship at the University of Glasgow, before moving to London in 2016. Chris is currently working on applications of Quantum Chromodynamics (the theory of quarks and gluons) to collider experiments, as well as how to relate this theory to gravity.

Acknowledgments

CDW is very grateful to Nadia Bahjat-Abbas, Andrés Luna, Stacey Melville, Ricardo Monteiro, Isobel Nicholson, Donal O'Connell, Alex Ochirov and Nicolas Westerberg for discussions and collaboration on related topics. He is supported by the UK Science and Technology Facilities Council (STFC).

References

- [1] Abbott BP, et al. (Virgo, LIGO Scientific). Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys Rev Lett.* 2016;116(6):061102.
- [2] Weinberg S. Ultraviolet divergences in quantum theories of gravitation. In: *General relativity: An einstein centenary survey; 1980.* p. 790–831.

- [3] Bern Z, Carrasco JJM, Johansson H. Perturbative Quantum Gravity as a Double Copy of Gauge Theory. *PhysRevLett.* 2010;105:061602.
- [4] Bern Z, Dennen T, Huang Yt, et al. Gravity as the Square of Gauge Theory. *PhysRev.* 2010;D82:065003.
- [5] Bern Z, Carrasco JJM, Johansson H. New Relations for Gauge-Theory Amplitudes. *Phys Rev.* 2008;D78:085011.
- [6] Dirac PAM. The quantum theory of the electron. *Proc Roy Soc Lond.* 1928;A117:610–624.
- [7] Noether E. Invariant Variation Problems. *Gott Nachr.* 1918;1918:235–257. [Transp. Theory Statist. Phys.1,186(1971)].
- [8] Feynman RP. Quantum theory of gravitation. *Acta Phys Polon.* 1963;24:697–722.
- [9] DeWitt BS. Quantum Theory of Gravity. 1. The Canonical Theory. *Phys Rev.* 1967; 160:1113–1148.
- [10] DeWitt BS. Quantum Theory of Gravity. 2. The Manifestly Covariant Theory. *Phys Rev.* 1967;162:1195–1239.
- [11] DeWitt BS. Quantum Theory of Gravity. 3. Applications of the Covariant Theory. *Phys Rev.* 1967;162:1239–1256.
- [12] Elvang H, Huang Yt. *Scattering Amplitudes.* 2013;.
- [13] Cheung C. *TASI Lectures on Scattering Amplitudes.* 2017;.
- [14] Bern Z, Carrasco JJM, Dixon LJ, et al. Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes. *Phys Rev.* 2012;D85:105014.
- [15] Bern Z, Davies S, Dennen T, et al. Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops. *Phys Rev.* 2015;D92(4):045041.
- [16] Monteiro R, O’Connell D. The Kinematic Algebra From the Self-Dual Sector. *JHEP.* 2011;07:007.
- [17] Badger S, Mogull G, Ochirov A, et al. A Complete Two-Loop, Five-Gluon Helicity Amplitude in Yang-Mills Theory. *JHEP.* 2015;10:064.
- [18] Saotome R, Akhoury R. Relationship Between Gravity and Gauge Scattering in the High Energy Limit. *JHEP.* 2013;1301:123.
- [19] Sabio Vera A, Serna Campillo E, Vazquez-Mozo MA. Color-Kinematics Duality and the Regge Limit of Inelastic Amplitudes. *JHEP.* 2013;1304:086.
- [20] Johansson H, Sabio Vera A, Serna Campillo E, et al. Color-Kinematics Duality in Multi-Regge Kinematics and Dimensional Reduction. *JHEP.* 2013;1310:215.
- [21] Johansson H, Sabio Vera A, Serna Campillo E, et al. Color-kinematics duality and dimensional reduction for graviton emission in Regge limit. 2013;.
- [22] Melville S, Naculich SG, Schnitzer HJ, et al. Wilson line approach to gravity in the high energy limit. *Phys Rev.* 2014;D89(2):025009.
- [23] Luna A, Melville S, Naculich SG, et al. Next-to-soft corrections to high energy scattering in QCD and gravity. *JHEP.* 2017;01:052.
- [24] Weinberg S. Infrared photons and gravitons. *Phys Rev.* 1965;140:B516–B524.
- [25] Naculich SG, Schnitzer HJ. Eikonal methods applied to gravitational scattering amplitudes. *JHEP.* 2011;05:087.
- [26] Laenen E, Stavenga G, White CD. Path integral approach to eikonal and next-to-eikonal exponentiation. *JHEP.* 2009;03:054.
- [27] White CD. Factorization Properties of Soft Graviton Amplitudes. *JHEP.* 2011;05:060.
- [28] Akhoury R, Saotome R, Sterman G. Collinear and Soft Divergences in Perturbative Quantum Gravity. *Phys Rev.* 2011;D84:104040.
- [29] Beneke M, Kirilin G. Soft-collinear gravity. *JHEP.* 2012;09:066.
- [30] Oxburgh S, White C. BCJ duality and the double copy in the soft limit. *JHEP.* 2013; 1302:127.
- [31] Johansson H, Nohle J. Conformal Gravity from Gauge Theory. 2017;.
- [32] Anastasiou A, Borsten L, Duff MJ, et al. Are all supergravity theories Yang-Mills squared?. 2017;.
- [33] Borsten L. On $D = 6$, $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (4, 0)$ theories. 2017;.
- [34] Cheung C, Shen CH, Wen C. Unifying Relations for Scattering Amplitudes. 2017;.

- [35] Kawai H, Lewellen D, Tye S. A Relation Between Tree Amplitudes of Closed and Open Strings. *NuclPhys.* 1986;B269:1.
- [36] Cachazo F, He S, Yuan EY. Scattering of Massless Particles: Scalars, Gluons and Gravitons. 2013;.
- [37] Cachazo F, He S, Yuan EY. Scattering of Massless Particles in Arbitrary Dimension. 2013;.
- [38] Cachazo F, He S, Yuan EY. Scattering Equations and KLT Orthogonality. 2013;.
- [39] Mason L, Skinner D. Ambitwistor strings and the scattering equations. *JHEP.* 2014; 1407:048.
- [40] Geyer Y, Lipstein AE, Mason LJ. Ambitwistor strings in 4-dimensions. *PhysRevLett.* 2014;113:081602.
- [41] Casali E, Geyer Y, Mason L, et al. New Ambitwistor String Theories. *JHEP.* 2015;11:038.
- [42] Casali E, Tourkine P. On the null origin of the ambitwistor string. *JHEP.* 2016;11:036.
- [43] Geyer Y, Mason L, Monteiro R, et al. Loop Integrands for Scattering Amplitudes from the Riemann Sphere. *Phys Rev Lett.* 2015;115(12):121603.
- [44] Geyer Y, Mason L, Monteiro R, et al. One-loop amplitudes on the Riemann sphere. *JHEP.* 2016;03:114.
- [45] Cheung C, Remmen GN. Twofold Symmetries of the Pure Gravity Action. *JHEP.* 2017; 01:104.
- [46] Bern Z, Grant AK. Perturbative gravity from QCD amplitudes. *Phys Lett.* 1999;B457:23–32.
- [47] Monteiro R, O’Connell D, White CD. Black holes and the double copy. *JHEP.* 2014; 1412:056.
- [48] Tangherlini F. Schwarzschild field in n dimensions and the dimensionality of space problem. *Nuovo Cim.* 1963;27:636–651.
- [49] Myers RC, Perry M. Black Holes in Higher Dimensional Space-Times. *Annals Phys.* 1986; 172:304.
- [50] Ridgway AK, Wise MB. Static Spherically Symmetric Kerr-Schild Metrics and Implications for the Classical Double Copy. 2015;.
- [51] Chong Z, Gibbons G, Lu H, et al. Separability and killing tensors in Kerr-Taub-NUT-de sitter metrics in higher dimensions. *PhysLett.* 2005;B609:124–132.
- [52] Taub AH. Empty space-times admitting a three parameter group of motions. *Annals of Mathematics.* 1951;53(3):pp. 472–490. Available from: <http://www.jstor.org/stable/1969567>.
- [53] Newman E, Tamburino L, Unti T. Emptyspace generalization of the schwarzschild metric. *Journal of Mathematical Physics.* 1963;4(7):915–923. Available from: <http://scitation.aip.org/content/aip/journal/jmp/4/7/10.1063/1.1704018>.
- [54] Luna A, Monteiro R, O’Connell D, et al. The classical double copy for TaubNUT space-time. *Phys Lett.* 2015;B750:272–277.
- [55] Coleman SR. Nonabelian Plane Waves. *PhysLett.* 1977;B70:59.
- [56] Aichelburg P, Sexl R. On the Gravitational field of a massless particle. *GenRelGrav.* 1971; 2:303–312.
- [57] Siegel W. Fields. 1999;.
- [58] Barnett SM. Maxwellian theory of gravitational waves and their mechanical properties. *New JPhys.* 2014;16:023027.
- [59] Adamo T, Casali E, Mason L, et al. Scattering on plane waves and the double copy. 2017; .
- [60] Luna A, Monteiro R, Nicholson I, et al. The double copy: Bremsstrahlung and accelerating black holes. 2016;.
- [61] Stephani H, Kramer D, MacCallum MA, et al. Exact solutions of Einstein’s field equations. 2003;.
- [62] Goldberger WD, Ridgway AK. Radiation and the classical double copy for color charges. *Phys Rev.* 2017;D95(12):125010.
- [63] Goldberger WD, Prabhu SG, Thompson JO. Classical gluon and graviton radiation from

- the bi-adjoint scalar double copy. 2017;.
- [64] Luna A, Monteiro R, Nicholson I, et al. Perturbative spacetimes from Yang-Mills theory. JHEP. 2017;04:069.
 - [65] Bjerrum-Bohr NEJ, Donoghue JF, Vanhove P. On-shell Techniques and Universal Results in Quantum Gravity. JHEP. 2014;02:111.
 - [66] Bjerrum-Bohr NEJ, Donoghue JF, Holstein BR, et al. Light-like Scattering in Quantum Gravity. JHEP. 2016;11:117.
 - [67] Anastasiou A, Borsten L, Duff MJ, et al. Yang-Mills origin of gravitational symmetries. Phys Rev Lett. 2014;113(23):231606.
 - [68] Anastasiou A, Borsten L, Duff MJ, et al. Twin supergravities from Yang-Mills theory squared. Phys Rev. 2017;D96(2):026013.
 - [69] White CD. Exact solutions for the biadjoint scalar field. Phys Lett. 2016;B763:365–369.
 - [70] De Smet PJ, White CD. Extended solutions for the biadjoint scalar field. 2017;.
 - [71] Wu TT, Yang CN. Some solutions of the classical isotopic gauge field equations. 1967;.
 - [72] Bern Z, Carrasco JJ, Dixon LJ, et al. Three-Loop Superfiniteness of N=8 Supergravity. Phys Rev Lett. 2007;98:161303.
 - [73] Bern Z, Carrasco JJ, Dixon LJ, et al. The Ultraviolet Behavior of N=8 Supergravity at Four Loops. Phys Rev Lett. 2009;103:081301.